

## Differentiation by the Chain Rule - Classwork

Suppose you were asked to take the derivatives of the following. Do so.

a)  $f(x) = (x^2 + 3x)^2$

b)  $f(x) = (x^2 + 3x)^3$

c)  $f(x) = (x^2 + 3x)^6$

d)  $f(x) = \sqrt{x^2 + 3x}$

a) causes no problem. b) is also not a problem but multiplying it out so you can take the derivative is a bit of a pain. You are capable of doing c) but clearly do not wish to. But d) can't be done with the knowledge you have.

We now introduce a method of taking derivatives of more complicated expressions. It is called the **chain rule**. If  $y = f(u)$  is a differentiable function of  $u$  and  $u = g(x)$  is a differentiable function of  $x$ , then  $y = f(g(x))$  is a differentiable function of  $x$  and  $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$  or equivalently,  $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

Example 1) If  $f(x) = (2x+5)^2$ , find  $f'(x)$  without and with the chain rule. Show they are equivalent.

a) without chain rule

b) with chain rule

Example 2) If  $f(x) = (2x+5)^3$ , find  $f'(x)$  without and with the chain rule. Show they are equivalent.

a) without chain rule

b) with chain rule

Example 3) If  $f(x) = (2x+5)^{10}$ , find  $f'(x)$

Example 4) If  $f(x) = \sqrt{2x+5}$ , find  $f'(x)$

Example 5) Find  $y'$  if  $y = \frac{1}{4x-3}$

Example 6) Find  $y'$  if  $y = (3x^2 - 2x + 1)^3$

Find the derivatives of the following:

$$7) y = (7 - 4x^2)^{\frac{2}{3}}$$

$$8) y = -5\sqrt{x^2 - 4x + 1}$$

$$9) y = \frac{-2}{\sqrt[4]{6x-1}}$$

More difficult problems: We now have 3 basic rules. Power rule, product rule, and quotient rule. Note that the chain rule is not a basic rule of differentiation. The chain rule is always in effect. Even when you find the derivative of  $y = 7x$ , your answer is 7 times the derivative of  $x$  which is  $7(1) = 7$ .

Find the derivatives of the following:

$$10) y = x^2(2x-3)^4$$

$$11) y = x\sqrt{4-x^2}$$

$$12) y = \left(\frac{2x-1}{2x+1}\right)^5$$

$$13) y = \frac{x}{\sqrt{x^2-1}}$$

$$14) y = \sqrt{\frac{x}{4x-1}}$$

$$15) y = \sqrt[3]{4x - \sqrt{2x+1}}$$

Given that  $f(2) = -3, f'(2) = 6, g(2) = 3, g'(2) = -2, f'(3) = 4$ , find the derivatives of the following at  $x = 2$ .

$$16) f(x) \cdot g(x)$$

$$17) \frac{f(x)}{g(x)}$$

$$18) [f(x)]^3$$

$$19) f(g(x))$$

## Differentiation by the Chain Rule - Homework

Find the derivatives of the following:

1.  $y = (3x - 8)^4$

2.  $y = (3x^2 + 2)^5$

3.  $y = 4(x^2 + x - 1)^{10}$

4.  $y = -5(4 - 9x)^{3/2}$

5.  $y = \frac{1}{3x - 2}$

6.  $y = \frac{-1}{(x^2 - 5x - 6)^2}$

7.  $y = \left(\frac{2}{2 - x}\right)^2$

8.  $y = \frac{4x}{(x + 1)^2}$

9.  $y = \frac{-3}{(x^3 - x^2 + 3)^3}$

10.  $y = x^3(5x - 1)^4$

11.  $y = \sqrt{1 - t}$

12.  $y = \sqrt[3]{3x^3 - 4x + 2}$

13.  $y = \frac{2}{\sqrt{2x + 3}}$

14.  $y = \frac{-1}{\sqrt{x + 1}}$

15.  $y = \sqrt{\frac{3x}{2x - 3}}$

16.  $y = \sqrt{x(1-2x)^2}$

17.  $y = \sqrt[3]{\frac{2t}{t^2-4}}$

18.  $y = (x^2 + 2x - 6)^2(1 - x^3)^2$

For each of the following, find the equation of the tangent line at the indicated point. Verify by calculator.

19.  $y = \sqrt{x^2 + 2x + 8}$  at  $(2, 4)$

20.  $y = \sqrt[5]{3x^3 + 4x}$  at  $(2, 2)$

21.  $y = \sqrt{\frac{3x-1}{2x+1}}$  at  $(-1, 2)$

Given the following information, find the value of the derivative of the functions at  $x = 3$ . Be careful, not all the information is needed to calculate these. Answers are next to the problem.

$x$	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
3	1	8	-3	-5
6	3	-2	4	5
8	-1	3	$\pi$	4
1	2	-6	5	0

22.  $f(x) + g(x)$  (Ans: -8)

23.  $f(x)g(x)$  (Ans: -29)

24.  $\frac{f(x)}{g(x)}$  (Ans:  $-\frac{19}{64}$ )

25.  $\frac{g(x)}{f(x)}$  (Ans: 19)

26.  $(f(x))^2$  (Ans: -6)

27.  $\frac{1}{g(x)}$  (Ans:  $\frac{5}{64}$ )

28.  $\sqrt{f(x)}$  (Ans:  $\frac{-3}{2}$ )

29.  $\sqrt{f(x) + g(x)}$  (Ans:  $\frac{-4}{3}$ )

30.  $f^3(x)g(x)$  (Ans: -77)

$x$	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
3	1	8	-3	-5
6	3	-2	4	5
8	-1	3	$\pi$	4
1	2	-6	5	0

31.  $\frac{1}{\sqrt[3]{g(x)}}$  (Ans:  $\frac{5}{48}$ )

32.  $\frac{f(x)}{f(x)+g(x)}$  (Ans:  $\frac{-19}{81}$ )

33.  $f(g(x))$  (Ans:  $-5\pi$ )

34.  $g(f(x))$  (Ans: 0)

35.  $f(f(x))$  (Ans: -15)

36.  $g(g(x))$  (Ans: -20)

37. The table below gives some values of the derivative of some function  $f$ . Complete the table by finding (if possible) the derivatives of each of the following transformations of  $f$ .

a)  $g(x) = f(x) - 2$

b)  $h(x) = 2f(x)$

c)  $r(x) = f(-3x)$

d)  $s(x) = f(2x+1)$

$x$	-2	-1	0	1	2	3
$f'(x)$	4	$\frac{2}{3}$	$-\frac{1}{3}$	-1	-2	-4
$g'(x)$						
$h'(x)$						
$r'(x)$						
$s'(x)$						